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Spacetime Quantization, Generalized Relativistic Mechanics, and Mach's Principle

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The introduction of an "elementary length" a *representing the ultimate limit for the smallest measurable distance leads to a generalization of Einstein's energy–momentum relation and of the usual Lorentz transformation. The value of* a *is left unspecified, but is found to be equal to* $hc/2E_u$, *where* E_u *is the total energy content of our universe. Particles of zero rest mass can only move at the velocity* c *of light in vacuum, while material bodies can move slower or faster than light, when* $a \neq 0$, *without violating the principle of causality. The laws of relativistic mechanics are actually generalized so that they include Mach's principle, since it is found that the universe as a whole can only be in a state of rest for any particular inertial observer.*

1. INTRODUCTION

Through the development of the theory of relativity and of quantum mechanics, we have learned that nature can impose some particular restrictions on our measurements, and that these restrictions are expressed by the existence of universal constants: the velocity of light in vacuum c and Planck's constant h. We should thus be prepared to consider also the possible existence of other restrictions, and in particular of an ultimate limit for the smallest measurable distance. The value a of this "elementary length" should actually be considered as an unknown quantity, while the usual theories are based on the implicit assumption that $a = 0$. These theories could correspond, indeed, to an approximation of a more general theory, including three finite, universal constants: c, h, and a.

It has been suggested quite often that there could exist a universal elementary length, corresponding to some combination of known universal constants, including, for instance, the electron charge e, the gravitational

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constant G, or the rest mass of some particular elementary particle. But we can only use the argument that the elementary length should be the smallest measurable distance. This leads directly to a relation between the value of a and the total energy content of our universe E_u.

A measurement of very small distances has to be performed, indeed, through some kind of scattering experiment. This means that the smallest measurable distance has to be of the order of the smallest possible wavelength λ. This value depends on the largest possible momentum, according to de Broglie's relation $p = h/\lambda$. But, to achieve the largest possible value of p, we have to use a massless particle of highest possible energy, so that $p = E/c$. Since it is absolutely impossible to consider a particle of higher energy than the total energy content of our universe E_u, we see that

$$a \sim hc/E_u \tag{1}$$

Nobody knows for sure if E_u is infinite. Many cosmological models are actually compatible with a finite value for E_u, and therefore with a finite value for a. We should thus try to find out if it is at least *possible* to construct a logically consistent theory that would take into account the restrictions imposed by the existence of an ultimate limit for the smallest measurable length, whatever its value may be.

Pauli[1] was convinced that "the spacetime is in need of a fundamental revision in the domain of very small dimensions." Nevertheless, he thought that it is not licit to introduce an elementary length, corresponding to a "cutoff" in the spectrum of possible wavelengths, for reasons of relativistic invariance. On the one hand, it would be necessary, indeed, to expect a universally constant value for this length, while this seems to be forbidden as a consequence of the Lorentz transformation for spacetime intervals.

This difficulty can be overcome, however, as will be shown in this article. The Lorentz transformation for the spacetime coordinates follows, indeed, from the requirement of relativistic invariance of some *differential equations*, like the Klein–Gordon equation. But these equations imply already the assumption that $a = 0$, since it would be physically meaningless to specify the variations of some function over space intervals that are smaller than a, or time intervals that are smaller than a/c when $a \neq 0$. This follows from the requirement that any physical law should be experimentally verifiable, at least in principle. We should thus start with a generalization of the usual laws, replacing all differential equations by *finite-difference equations* for the variation of any function in space and time. The relativistic invariance of this theory allowing now for $a \neq 0$ should then be verified in a second step. Instead of using the usual Lorentz transformation to reject the possible existence of a finite elementary length, we use then the concept of such a length to show that the usual Lorentz transformation can be generalized.

2. LAWS OF MOTION FOR A PARTICULAR INERTIAL REFERENCE FRAME

2.1. Basic Assumptions

I. The origin and the orientation of the spatial reference axes can be chosen arbitrarily. This depends only on our chosen experimental setup for measuring coordinates.

II. An ideally exact measurement of the values of the spacetime coordinates could be performed in principle by a juxtaposition of smallest measurable distances along the chosen reference axes. This means that the eigenvalues of the spacetime coordinates (x, y, z, ct) correspond to a spacetime lattice of lattice constant a, depending on the chosen frame of reference.

III. Any field that is associated with a given kind of particle can only be defined on the spacetime lattice points where a particle could be observed with some given probability. This is true even for macroscopic bodies, as soon as we define some particular point attached to this body, like its center of mass, to localize it exactly.

IV. The variation of the fields in space and time have to be specified by finite-difference equations obtained by generalizing the usual differential equations through the following correspondence principle for second partial derivatives:

$$\partial_x^2 f(x) \leftrightarrow D_x^2 f(x) = \frac{f(x+a) + f(x-a) - 2f(x)}{a^2} \tag{2}$$

We have chosen this rule, since it is the simplest one, resulting from the fact that we can only consider intervals Δx that are integer multiples of the quantum of length a. It is interesting to note that Heisenberg[2] used a similar correspondence principle for the initial formulation of quantum mechanics. The derivative $\nu_i = \partial E/\partial J_i$ had to be replaced, indeed, by a finite derivative, since the energy E is a function of action integrals J_i, which are quantized, so that ΔJ_i is always an integer multiple of Planck's constant h.

The concept of a spacetime quantization and of finite-difference equations instead of the usual differential equations has been considered already by Ambarzumian and Iwanenko[3] and Ruark.[4] Meessen[5] used the correspondence principle given above to get a generalized Klein–Gordon equation, leading to a modification of Einstein's energy–momentum relation. The internal consistency of such a theory was checked[6] by considering the limiting case where $c = \infty$, since three different definitions of the average velocity of a particle should then yield equivalent expressions. The concept of field quantization can also be extended,[7] as well as Dirac's equation, which leads to the appearance of new degrees of freedom.[8]

2.2. Generalization of Einstein's Energy–Momentum Relation

The possible states of motion of a free particle of given rest mass m_0 in a particular inertial reference frame are defined by the possible values of the energy–momentum variables $(E/c, p_x, p_y, p_z)$. Einstein showed that these values are connected by the following equation:

$$(E/c)^2 - \sum_i p_i^2 = (m_0 c)^2 \tag{3}$$

De Broglie discovered, on the other hand, that we can associate a plane wave

$$\psi(x, y, z, ct) = A \exp[i(k_x x + k_y y + k_z z - \omega t)] \tag{4}$$

to any particle of well-defined energy and well-defined momentum by means of the relations

$$E = \hbar\omega \qquad \text{and} \qquad p_i = \hbar k_i \tag{5}$$

where $i = x, y, z$. The energy–momentum relation (3) is then interpreted as a dispersion relation, resulting from the fact that ψ has to verify the Klein–Gordon equation

$$\left(\sum_i \partial_i^2 - \partial_{ct}^2\right) \psi = (m_0 c/\hbar)^2 \psi \tag{6}$$

With the correspondence principle (2), we now get the "generalized Klein–Gordon Klein"

$$\left(\sum_i D_i^2 - D_{ct}^2\right) \psi = (m_0 c/\hbar)^2 \psi \tag{7}$$

This equation is still satisfied by functions of the form (4), although we consider the spacetime coordinates as being quantized in terms of the elementary length a. But we get now the generalized energy–momentum relation

$$\sin^2(Ea/2\hbar c) - \sum_i \sin^2(p_i a/2\hbar) = (m_0 ca/2\hbar)^2 \tag{8}$$

This relation reduces to (3) when $a = 0$ or when E/c, p_i, and $m_0 c \ll \hbar/a$. The periodicity of the energy momentum relation allows us to restrict our considerations to the first Brillouin zone, where

$$-\pi\hbar/a < E/c, p_x, p_y, p_z < +\pi\hbar/a$$

All other (real) values of the energy–momentum variables would not only reproduce the same relation (8), but also the same values for the amplitude of the function (4) on the chosen spacetime lattice. The variation of this function between the lattice points is indeed irrelevant, as it is well known for the analogous behavior of sound waves in a crystal.

2.3. The Largest Possible Rest Mass

Requiring that the values of the energy–momentum variables should be real, it follows from (8) that m_0 can take any value between 0 and $M = 2\hbar/ca$. A body whose rest mass would be equal to M could only be in a state of rest ($p_i = 0$), while its energy would have the largest possible value, defined by

$$E/c = \pi\hbar/a \qquad \text{or} \qquad E = E_u = hc/2a$$

We have to interpret the energy E_u as being equal to the total energy content of our universe, since a body of such a large rest mass that its rest-energy would be equal to the total energy content of our universe would necessarily have to be at rest. There is indeed no energy that is left over and that could appear in the form of kinetic energy for such a body. The essential content of the generalized energy–momentum relation (8) corresponds thus to the fact that our usual laws have certainly to be modified in the domain of extremely high energies comparable to the total energy content of our universe when this energy $E_u \neq \infty$.

2.4. Generalization of Newton's Second Law of Motion

The idea of a spacetime lattice was actually introduced in 1870, by Clifford,[9] who questioned the belief that there has necessarily to exist a spacetime continuum, by imagining "discontinuous motions." This means that all particles were supposed to "jump" from one lattice point to another lattice point. This model shows that the concepts of a spacetime continuum is actually derived from the concept of continuous existence of a moving particle. In order to get from one point to another point, it is then necessary, indeed, that the particle passes through a continuous sequence of intermediate points.

It is not very convincing, of course, to require a discontinous existence of moving particles. But this is not necessary, when we take into account the probabilistic interpretation of quantum mechanical wave functions. We have merely to consider the motion of a wave packet, defining a progressively changing probability distribution for the particle on various spacetime lattice points.

The deterministic classical motion then corresponds to the average motion of the particle, in the quantum mechanical sense, even when the spacetime coordinates are quantized. Since the wave packet is formed by a superposition of plane waves of type (4), with a given distribution for the possible values of the momentum components p_i, we can define the average motion by considering the point where all components are in phase. The resulting expression for the components of the average velocity of the particle corresponds thus to the usual definition of the group velocity:

$$v_i = \partial\omega/\partial k_i = \partial E/\partial p_i \tag{9}$$

The values of the derivatives have to be taken for the average values of the momentum components, which will be simply designated by p_i. It should be noted that the average position of the particle can be situated between the lattice points, since it is defined by a calculation and not by a direct, single measurement.

The function $v_i = v_i(p_1, p_2, p_3)$ allows us to predict the components of the acceleration of the particle when the momentum components vary with time:

$$a_i = \dot{v}_i = \sum_j (\partial v_i/\partial p_j)\dot{p}_j = \sum_j (\partial^2 E/\partial p_i\, \partial p_j)\dot{p}_j$$

We assume here that the variations are sufficiently slow, so that we make no appreciable error by considering ordinary time derivatives, represented by dots. This leads to a generalization of Newton's law of motion:

$$a_i = \sum_j F_j/m_{ij} \tag{10}$$

$$F_j = \dot{p}_j \qquad \text{and} \qquad 1/m_{ij} = \partial^2 E/\partial p_i\, \partial p_j \tag{11}$$

The acceleration is thus determined by an external factor (F_j) and an internal factor (m_{ij}). The relations (9)–(11) reduce to the usual laws of special relativity when we make use of Einstein's energy–momentum relation (3). It is necessary, of course, to distinguish the (scalar) momentum mass $m = p_i/v_i$ from the inertial mass tensor, which implies the existence of a transverse and a longitudinal mass with respect to the direction of motion of the particle.

2.5. Generalization of Relativistic Mechanics

We have now to consider the physical meaning of the relations (8)–(11), which take into account the existence of the constants c, h, and a. To see more clearly the new features, we restrict our considerations to a particle moving along one of the chosen reference axes. This allows us to set $p = p_x$ with $p_y = p_z = 0$, while (8) reduces to

$$\sin^2 (Ea/2\hbar c) - \sin^2 (pa/2\hbar) = (m_0 ca/2\hbar)^2 \tag{12}$$

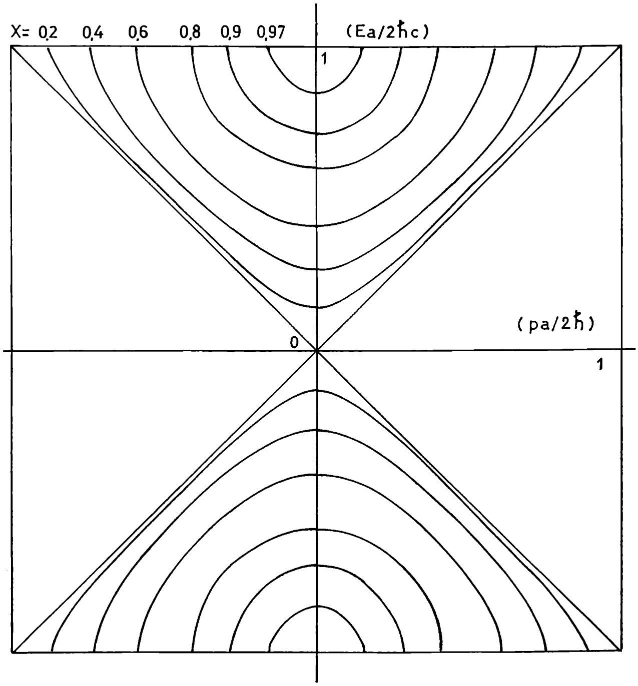

Fig. 1. Graphical representation of the generalized energy–momentum relation in the first Brillouin zone for different values of the rest mass ($x = m_0ca/2\hbar$).

This corresponds to a "deformed hyperbola" that is "cut off" at the boundary of the first Brillouin zone, so that $| E/c |$ and $| p | < \pi\hbar/a$, as indicated in Fig. 1. The extended zone scheme would correspond to a juxtaposition of "tiles" bearing the drawing of Fig. 1. The solid lines in Fig. 1 correspond to possible states of motion of material bodies of various rest masses. For small values of m_0 the curve approaches the straight lines, which represent the energy–momentum relation $E^2 = c^2p^2$ for a particle of zero rest mass. For large values of m_0 the curve shrinks toward the limiting points $E/c = \pm h/2a$ and $p = 0$. We mentioned that the upper point corresponds to the only state of motion of the universe as a whole. Negative energy states only have a meaning, of course, in Dirac's hole theory.

The energy–momentum relation for a material body of given rest mass $m_0 \neq 0$ is redrawn in Fig. 2a for the upper right quarter of the first Brillouin zone, with an indication of Newton's classical approximation (C) and Einstein's relativistic approximation (R). In Fig. 2a we represent the corresponding variations of the velocity $v = v(p)$. This curve cuts the level $v = c$ at a finite value for p, corresponding to an energy $E > E_u/2$. Beyond

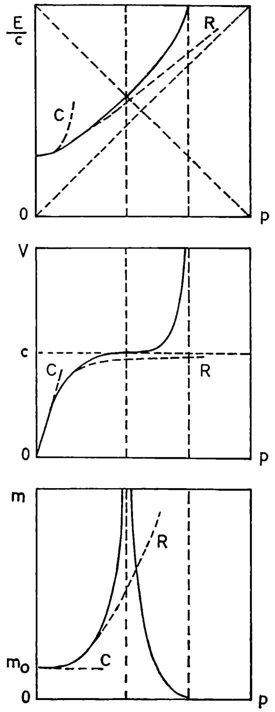

Fig. 2. (a) Variation of the energy E vs. momentum p for a particle of given rest mass in the upper quadrant of the first Brillouin zone. The classical and relativistic approximations are represented by the curves labeled C and R. (b) Variation of the (average) velocity v vs. (average) momentum p. (c) Variation of the longitudinal inertial mass m vs. momentum p.

this point the curve increases again more and more rapidly, so that $v \to \infty$ when $E \to E_u$. This is only true, however, for ordinary material bodies. When $m_0 = 0$, we get $v = c$ for any value of p. For $m_0 = M = 2\hbar/ca$, the velocity v cannot be defined any more, but we can speak of a state of absolute rest, since $p \equiv 0$.

Figure 2c represents the variation of the longitudinal inertial rest mass $m = m(p)$. We see that $m \to \infty$ when $v \to c$, as in special relativity, but it is now possible to reach ultra-light velocities ($v > c$). The inertial mass decreases when $v \to \infty$. The "light barrier" cannot be overcome, however, by accelerating a particle through a continuous application of a given force, since the acceleration $a \to 0$ when $v \to c$. It is only possible to "jump over this barrier" by means of a quantum mechanical transition where the energy is suddenly increased or decreased, for instance, by the absorption or emission of photons.

The conclusion that a material body can move faster than light when one does not assume a priori that $a = 0$ is at least of considerable importance for a better understanding of the special theory of relativity. This theory is only based, indeed, on the requirement that the velocity of light in vacuum should be a universal constant and does not basically impose a limit for the highest possible velocity of any material body. It should also be noted that the present theory implies that the universe cannot be composed of matter and antimatter in a symmetric way, so that the total amount of energy in our universe would be zero. Although it is possible to create antimatter by exciting particles from negative to positive energy states, there exists only a total excitation energy equal to E_u.

The analytical relation for $v = v(p)$ can be obtained immediately by derivation of (10). This leads to

$$(v/c)\sin(Ea/\hbar c) = \sin(pa/\hbar) \tag{13}$$

showing that $v = c$ when $E = cp$, while the "frontier" where material bodies are passing from infra-light velocities to ultra-light velocities is defined by the condition

$$\frac{Ea}{\hbar c} = \pi - \frac{pa}{\hbar} \quad \text{or} \quad \frac{E}{c} = \frac{\pi\hbar}{a} - p \tag{14}$$

which is represented by the second diagonal in Fig. 2a. Deriving (13) with respect to p, we get a relation for the inertial mass m:

$$m[\cos(pa/\hbar) - (v/c)^2\cos(Ea/\hbar c)] = M\sin(Ea/\hbar c)$$

2.6. Mach's Principle

Newton realized already that the law of motion $F = ma$ implies the existence of a particular set of reference frames, called inertial reference frames, since these are the only frames where this law and the principle of inertia are valid for true forces. The acceleration would appear to be different, indeed, for any reference frame that is accelerated with respect to the former ones. This leads to the appearance of fictitious forces, resulting from an acceleration of the reference frame, while true forces are due to actual interactions of the observed body with other bodies. Newton thought that there exists an absolute space, that remains always motionless by its very nature, and that inertial reference frames are frames that are at rest or in a state of uniform motion with respect to this absolute space.

Mach[10] objected, as had Berkeley, that absolute space is a purely "metaphysical concept" beyond the reach of experimental verification. Since it is only meaningful to define the motion of one body with respect to some other body, he suggested that "absolute space" should be replaced by the most embracing system, i.e., the universe as a whole. This is called Mach's principle, since it had to be postulated in addition to Newton's law of motion.

The present theory removes this artifact, since we can derive both laws from the same basic principles. Defining an inertial reference frame as being a frame where the generalized Klein–Gordon equation (7) and the resulting energy–momentum relation (8) are valid, we were led to the generalization (10) of Newton's law and to the conclusion that the universe as a whole would be at rest in this system.

3. RELATIVISTIC INVARIANCE

3.1. The Generalized Lorentz Transformation

Any physical law represents a statement about a physical system that is independent of the chosen reference frame. Nevertheless, it is necessary to express this law in terms of a mathematical relation, connecting various results of measurement that will be obtained by performing measurements with respect to a given frame of reference. The principle of special relativity requires simply that there exists a correspondence between the results of measurements that are performed in different inertial reference frames, so that any physical law expresses an intrinsic property of the system, independent of the chosen reference frame.

Let us consider therefore two different inertial frames of reference where a measurement of the energy–momentum variables of a given body would yield different results: $(E/c, p_x, p_y, p_z)$ and $(E'/c, p_x', p_y', p_z')$. By measuring these values for all possible states of motion of the body, we have to get the same energy–momentum relation in both inertial frames of reference. This leads with (8) to the requirement that

$$\sin^2(Ea/e\hbar c) - \sum_i \sin^2(p_i a/2\hbar) = \sin^2(E'a/2\hbar c) - \sum_i \sin^2(p_i a/2\hbar) = K$$

where the constant K is in general different from zero. Assuming that the inertial frames of reference are moving relative to one another along a common reference axis, we set

$$p_y' = p_y \qquad \text{and} \qquad p_z' = p_z \tag{15}$$

With $p' = p_x'$ and $p = p_x$, we can achieve the required invariance by means of the transformation

$$\sin(E'a/2\hbar c) = \gamma[\sin(Ea/2\hbar c) - \beta \sin(pa/2\hbar)] \tag{16}$$

$$\sin(p'a/2\hbar) = \gamma[\sin(pa/2\hbar) - \beta \sin(Ea/2\hbar c)] \tag{17}$$

where

$$\gamma = (1 - \beta^2)^{1/2} \tag{18}$$

The parameter β specifies the state of relative motion of the chosen frames of reference. Its value has to be determined by considering the motion of some (arbitrarily chosen) reference object with respect to both frames. It follows indeed from (17) that this body will appear to be at rest in one reference frame ($p' = 0$) when E and p are related in the other reference frame so that

$$\beta = \frac{\sin(pa/2\hbar)}{\sin(Ea/2\hbar c)} \tag{19}$$

When we assume that the reference body is at rest with respect to the y and z axes ($p_y = p_z = 0$), we can conclude from (12) and (13) that

$$\beta = \left[1 - \frac{(m_0 ca/2\hbar)^2}{\sin^2(Ea/2\hbar c)}\right]^{1/2} \quad \text{or} \quad \beta = \frac{v}{c}\frac{\cos(Ea/2\hbar c)}{\cos(pa/2\hbar)} \tag{20}$$

The first relation shows that β depends in general on the energy and the rest mass of the chosen reference body. This had to be expected, since the universe as a whole has to be in a state of rest for any particular inertial reference frame (the corresponding value of β is zero). On the other hand, it follows from (19) and (12) that $\beta < 1$ when $m_0 \neq 0$ even when the reference body moves at ultra-light velocities, so that (18) remains always real.

The second relation (20) shows that $\beta = v/c$ for E/c and $p \ll \hbar/a$, or for any value of E and p when $a = 0$. This means that the mass of the reference body is irrelevant in our usual theory, and that we are then allowed to consider v as the relative velocity of the reference frames.

3.2. The Velocity Addition Law and the Doppler Effect

The generalized Lorentz transformation also allows to get a relation between the (average) velocities $v = dE/dp$ and $v' = dE'/dp'$ of a particle in both reference frames. For simplicity we assume again that this particle moves only along the common reference axis of both reference frames. Deriving (16) and (17) with respect to p and taking the ratio of the resulting expressions, we get

$$V' = \frac{V - \beta c}{1 - \beta V/c} \tag{21}$$

where β is the parameter that specifies the relative motion of the chosen reference frames, while

$$V = v \frac{\cos(Ea/2\hbar c)}{pa/2\hbar} = c \frac{\sin(pa/2h)}{\sin(Ea/2\hbar c)} \tag{22}$$

with a similar expression for V'. We see that $V = v$ and $V' = v$ when E/c and $p \ll \hbar/a$. In general, it is necessary, however, to replace the velocities v and v' by the "quasivelocities" V and V' which are related by the addition law for velocities in the usual theory. We note that $\beta = V/c$ when $V' = 0$, in agreement with (19) and (20).

We see that the velocity of light in vacuum is a universal constant for all inertial observers, since $m_0 = 0$ implies that $E = cp$ and $V = v = c$, according to (13) and (22), while (21) leads to $V' = v' = c$, for any value of β. Einstein's expression[11] for the relativistic Doppler effect can also be generalized, even for sources that move faster than light. For photons, ($E = \hbar\omega = cp$ and $E' = \hbar\omega' = cp'$) it follows from (17) that

$$\sin(\omega' a/2c) = [(1 - \beta)/(1 + \beta)]^{1/2} \sin(\omega a/2c)$$

3.3. A Change of Representation for the Spacetime Variables

The usual Lorentz transformation for the spacetime variables results from the requirement of relativistic invariance of the differential equation (6). In the particular case of two inertial reference frames that move relative to one another along the x and x' axes, we can satisfy this condition by means of the well-known transformation

$$x' = \gamma(x - \beta ct), \qquad ct' = \gamma(ct - \beta x), \qquad y' = y, \qquad z' = z \tag{23}$$

with (18). Schild,[12] Hill,[13] and Ahmavaara[14] tried to generalize the Lorentz transformation for the spacetime coordinates when there exists a "discrete structure" of space and time, by considering the subgroup of the usual Lorentz transformations, called "integral Lorentz transformations," which correspond to rotations that superpose (x, y, z, ict) and (x', y', z', ict') lattices on each other. This procedure is not physically acceptable, for the following reasons: (i) The choice of the inertial reference frames should be completely arbitrary, instead of being restricted to a discrete ensemble. Moreover, it is not necessary (ii) to assume the existence of an absolute spacetime lattice and (iii) to require a deterministic correspondence between the possible values of the spacetime coordinates.

We have actually to consider a change of representation of the state of motion of the particle, in terms of the quantized spacetime coordinates, which represent the only possible eigenvalues that could be found by an ideally exact measurement of these coordinates in the chosen frames of reference. An exact knowledge of the values (x, y, z, ct) of these coordinates in one frame would thus only allow for a statistical prediction of the distribution of the results of measurement for the (x', y', z', ct') coordinates in the other frame. For each one of the reference axes, the problem is actually similar to the well-known situation for the components of a given angular momentum vector when these components are measured along two different reference axes.

It is one of the basic features of quantum mechanics that a given state of motion can be expressed equally well by means of a spacetime representation as by an energy–momentum representation. For a particle moving along the x axis we then get the expression

$$\psi(x, ct) = \int c(p) e^{i(px-Et)/\hbar}\, dp$$

The deterministic law of correspondence (16)–(18) allows us therefore to define the transformed function

$$\psi'(x', ct') = \int c(p')\, e^{i(p'x'-E't')/\hbar}\, dp'$$

For all practical purposes it is actually sufficient to use the generalized Lorentz transformation for the energy–momentum variables.

3.4. Average Coordinates and Causality

Although the actual values of the quantized spacetime coordinates can only be predicted statistically, there exists a well-defined average position for the particle in each one of the chosen frames of reference. When the particle moves uniformly, we can say that

$$\Delta\langle x\rangle = v\,\Delta t \qquad \text{and} \qquad \Delta\langle x'\rangle = v'\,\Delta t'$$

where t and t' are the quantized time variables, while v and v' are the group velocities, which are related to one another by (21) and (22), so that

$$\frac{\Delta\langle x'\rangle}{c\,\Delta t'}\,\frac{\cos(E'a/2\hbar c)}{\cos(p'a/2\hbar)} = \frac{\Delta\langle x\rangle[\cos(Ea/2\hbar c)/\cos(pa/2\hbar)] - \beta c\,\Delta t}{c\,\Delta t - \beta\,\Delta\langle x\rangle\cos(Ea/2\hbar c)/\cos(pa/2\hbar)}$$

When the time intervals Δt and $\Delta t'$ are sufficiently large, we can consider them also as continuous variables, and replace the usual Lorentz transformation (23) by

$$\langle x'\rangle\,\frac{\cos(E'a/2\hbar c)}{\cos(p'a/2\hbar)} = \gamma\left[\langle x\rangle\,\frac{\cos(Ea/2\hbar c)}{\cos(pa/2\hbar)} - \beta ct\right] \tag{24}$$

$$ct' = \gamma\left[ct - \beta\langle x\rangle\,\frac{\cos(Ea/2\hbar c)}{\cos(pa/2\hbar)}\right] \tag{25}$$

To show the physical meaning of these relations, we consider a particle (or a signal) that is created at a given instant and annihilated at some later instant. During the time Δt of its existence it moves a distance $\langle x \rangle = v\,\Delta t$ in one reference frame. It follows then from (25) and (22) that another observer will get the value

$$\Delta t' = \gamma(1 - \beta V/c)\,\Delta t \tag{26}$$

when he measures the time of existence of the particle. Since $\beta < 1$ and $V < c$ while $\gamma > 0$, we see that $\Delta t'$ is always positive when Δt is positive. The principle of causality, requiring that the sequence of events should be preserved for all inertial observers, is thus verified, even when the particle moves faster than light.

This is an important conclusion, since the principle of causality cannot be satisfied within the framework of the usual theory of relativity for particles that move faster than light, i.e., for tachyons.[15] These hypothetical particles satisfy Eq. (3) with a purely imaginary rest mass. It follows then from (23), with $\Delta x = v\,\Delta t$, that

$$\Delta t' = \gamma(1 - \beta v/c)\,\Delta t$$

$\Delta t'$ and Δt can now have different signs, since $v > c$.

3.5. The Energy–Momentum Horizon

We mentioned that the generalization (8) or (12) of Einstein's relation leads to a periodicity in the space of the energy–momentum variables, so that we can restrict our considerations to the first Brillouin zone (Fig. 1). This means that we take into account increasing values of E/c and p, up to the point where the sine functions become equal to one. It is possible, however, to consider larger values for the sine functions, when the values of E/c and p are purely imaginary. The functions (4) are then attenuated, and the values E/c and p do not correspond any more to possible results of measurement for the energy–momentum variables. But we get, nevertheless, acceptable solutions of the general equations, defining "forbidden states," in the sense of forbidden energy bands for electrons in solid state physics.

These solutions have to be considered, indeed, when we represent the generalized energy–momentum relation (12) by a hyperbola for the sine functions (solid lines in Fig. 3), and when we represent the generalized Lorentz transformation (16) and (17) as being equivalent to a change of reference axes (broken lines in Fig. 3). The so called "allowed" states of motion, with measurable values for the energy–momentum variables, correspond then to those values of the sine functions that are less than or equal to one. We see that this corresponds to a different "horizon" for two different inertial frames of reference (square and parallelogram in Fig. 3).

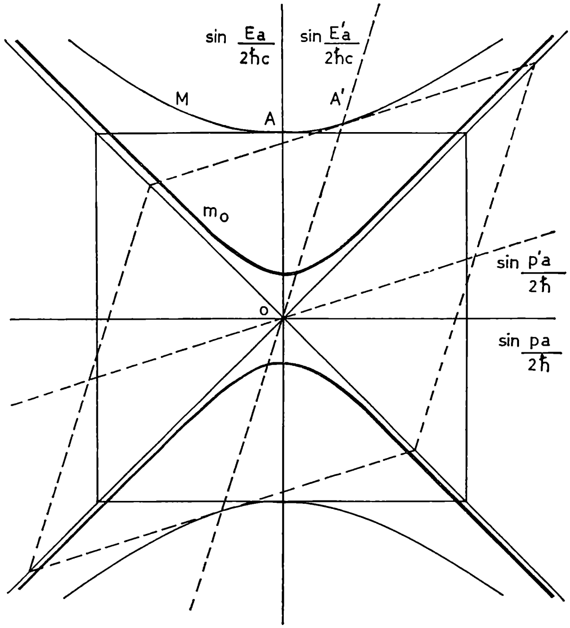

Fig. 3. Graphical representation of the generalized energy–momentum relation and the generalized Lorentz transformation for two inertial frames of reference. The square and the parallelogram represent the horizons separating the allowed from the forbidden states of motion for each reference frame. Within these horizons, there exists a deterministic correspondence between the results of measurements for energy–momentum variables.

To interpret these results, we have to recall that the theory of spacetime quantization is a "quantum theory" and that the predicted spectrum of possible measurement results is always defined with respect to a given experimental setup. Thus we do not always get a continuous spectrum of possible eigenvalues, allowing us to prescribe a deterministic law of correspondence between the results of measurement in two different frames of reference, as was customary in classical physics. Some results of measurement are forbidden for each frame of reference.

The fact that the horizon for the observable energy–momentum variables is different for different inertial observers is actually required by our formulation of Mach's principle. The energy–momentum relation for the body of largest possible rest mass M corresponds indeed to a hyperbola in Fig. 3. But there exists only one point (A or A') of this hyperbola which touches the horizon for two different frames of reference, and which corresponds always to the state of rest ($p = 0$ and $E = E_u$ or $p' = 0$ and $E' = E_u$). We can actually draw the same graph (Fig. 1) for the energy–momentum variables in each of the reference frames.

4. CONCLUSIONS

The concept of an "elementary length" can only be introduced in our physical theories within the frame work of quantum mechanical concepts. This appears already in the definition (1) of the order of magnitude of the quantum of length.

Another argument corresponds to the fact that distances are only quantized along the chosen reference axes. This solves the old problem, raised by Pythagoras, who thought that any line should be considered as being constructed by means of a juxtaposition of very small, but finite and indivisible line elements. He was therefore very disapointed when he discovered that the diagonal and the side of a square are incommensurable: their ratio thus defined an "irrational" number. This is not relevant, however, when the quantization is only associated with actual distance measurements, performed along the three chosen reference axes for a three-dimensional space, while all other distances are simply determined by calculation.

This concept allows also the preservation of the basic homogeneity and isotopy of space, since the reference axes can be chosen arbitrarily. We get thus a different spacetime lattice for all different frames of reference. There is no need for an absolute spacetime lattice, as seemed necessary before the advent of quantum mechanics.

Moreover, there is no need for considering "discontinuous motions," since a particle does not have to cease to exist when it "jumps" from one lattice point to another one. It is sufficient to consider a progressive change of the probability distribution.

Finally, it is sufficient to define the possible states of motion in the energy–momentum representation to define the average velocity and the inertial mass of the particle and to describe the effect of a change of reference frame in a simple, deterministic way.

Replacing the usual differential equation by finite–difference equations, to take into account arbitrarily small intervals of space and time when $a \neq 0$, we also had to replace the laws of relativistic mechanics by more general ones. Although the differences appear only at energies that are comparable to the total energy content of our universe E_u, we arrived at two important consequences: (i) Material bodies can move faster than light when $a \neq 0$, and (ii) inertial frames are those frames where Newton's second law is valid for true forces and where a body that has the highest possible rest energy E_u would be at rest.

The actual value of the quantum of length a can only be determined by measurements. It would actually be sufficient to show that the total energy content of our universe $E_u \neq \infty$ to be sure that $a \neq 0$. For the moment we can only show that such an assumption is not illogical and that it even has some very attractive features, like the justification of Mach's principle or a justification of the "cutoff" procedure which is sometimes used to avoid divergences appearing in our present field theories. Some preliminary results encourage us to think that the theory of spacetime quantization might also provide a natural explanation of the new quantum numbers characterizing elementary particles. It is already very instructive, however, to get a deeper understanding of our present theories by considering them as a particular case of a more general theory.

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